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SOME ASYMPTOTIC *i*th RAMSEY NUMBERS

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Let G be a graph with chromatic number $\chi(G)$ and let $t(G)$ be the minimum number of vertices in any color class among all $\chi(G)$-vertex colorings of G. Let H' be a connected graph and let H be a graph obtained by subdividing (adding extra vertices to) a fixed edge of H'. It is proved that if the order of H is sufficiently large, the ith Ramsey number $r_i(G, H)$ equals $[((\chi(G)-1)(|H|-1)+t(G)-1)/i]+1$.

In this paper, all graphs considered will be simple (no loops or multiple edges), undirected and finite. If G is a graph, $V(G)$ will denote its vertex set, $E(G)$ its edge set, uv the edge with endpoints u and v, and we adopt the convention that $|V(G)|=|G|$. As usual, K_n will denote a complete graph on n vertices, P_n a path on n vertices, C_n a cycle on n vertices, $K(n_1,\ldots,n_k)$ a complete k-partite graph with parts of size $n_1,\ldots,n_k$ respectively, and $[x]$ will denote the greatest integer less than or equal to x. For graphs G and H, we write $G\subset H$ to indicate that G is isomorphic to a subgraph of H. Notation not specifically mentioned will follow that in Bondy and Murty [4].

A graph G is said to have a *factorization* into the factors F_1 and F_2 (written $G=F_1\oplus F_2$) if (i) $V(F_1)=V(F_2)=V(G)$, (ii) $E(F_1)\cup E(F_2)=E(G)$ and (iii) $E(F_1)\cap E(F_2)=\emptyset$. The factorization $G=F_1\oplus F_2$ is called a *good factorization* with respect to (G_1, G_2) provided $G_1\not\subset F_1$ and $G_2\not\subset F_2$. In addition, we write $G\to(G_1, G_2)$ to mean if $G=F_1\oplus F_2$ is an arbitrary factorization of G, then $G_1\subset F_1$ or $G_2\subset F_2$.

Let $K(i;p)$ denote the complete p-partite graph whose parts each contain i vertices. Benedict [1] defines the *ith Ramsey number* $r_i(G, H)$ for graphs G and H to be the least positive integer p such that $K(i;p)\to(G, H)$. In particular, $r_1(G, H)$ is the generalized Ramsey number $r(G, H)$.

Although very little is known about $r_i(G, H)$ for $i\geqslant 2$, we will evaluate $r_i(G, H)$ for some familiar graphs G and H where $|H|$ is sufficiently large. In order to obtain a rather general result, let G be a graph with chromatic number $\chi(G)$ and let $t(G)$ be the minimum number of vertices in any color class among all $\chi(G)$-vertex colorings of G. Let H' be a connected graph and let H be a graph obtained by subdividing (adding extra vertices to) a fixed edge of H'. Then if $|H|$ is large enough, we are able to determine $r_i(G, H)$.

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In particular, it is clear that $r_i(K_m, K_n) = r(K_m, K_n)$, but we shall show that the situation is different for $r_i(K_m, P_n)$ and $r_i(K_m, C_n)$ whenever n is very large. For example, the ith Ramsey number always satisfies

$$[(r(G, H)-1)/i]+1 \leq r_i(G, H) \leq r(G, H)$$

and therefore $r_i(K_m, K_n)$ equals the upper bound. On the other hand, $r_i(K_m, P_n)$ and $r_i(K_m, C_n)$ equal the lower bound for sufficiently large n. This may be due to the relatively few edges contained in paths and cycles.

First, however, we need the following definitions. Let G be a graph. If G_1 and G_2 are subgraphs of G, then the intersection $G_1 \cap G_2$ of G_1 and G_2 is the subgraph with vertex set $V(G_1) \cap V(G_2)$ and edge set $E(G_1) \cap E(G_2)$. If V' is any subset of $V(G)$, the set of all vertices adjacent in G to vertices in V' will be denoted by $N_G(V')$. In particular, if $V' = \{v\}$, we write $N_G(v)$ for $N_G(\{v\})$. We also write $d_G(v) = |N_G(v)|$. In addition, the subgraph of G induced by V' will be denoted by $G[V']$; i.e., $G[V']$ is the subgraph of G whose vertex set is V' and whose edge set is the set of those edges of G that have both endpoints in V'. The subgraph $G[V(G)-V']$ will be denoted by $G-V'$ and we will write $G-v$ for $G-\{v\}$.

We shall make extensive use of Turán's theorem throughout this paper, so we begin by stating a version of this famous theorem. We also include a useful corollary.

Theorem 1 (Turán [10]). *Let G be a graph on m vertices which contains no K_n. Then*

$$|E(G)| \leq \frac{(n-2)(m^2-t^2)}{2(n-1)} + \binom{t}{2},$$

where $m \equiv t \pmod{n-1}$ and $0 \leq t < n-1$.

Corollary 2. *If G is a graph with minimum degree $\delta(G) \geq |G|-i$ for some positive integer i, then G contains a complete graph on $[|G|/i]$ vertices.*

Next we generalize a result due to Burr [5]. The proof makes use of the well known fact that

$$r(K_m, K_n) \leq \binom{m+n-2}{m-1} \qquad \text{(Erdös and Szekeres [8]).}$$

Theorem 3. *Let i be a positive integer, and let G and H' be fixed graphs. Let $ab \in E(H')$, and let H be a graph obtained from H' by subdividing ab such that $|H| \geq |H'| + |G|^2 i$ and*

$$\binom{|G|+|H'|+|G|^2 i-2}{|G|-1} \leq [((\chi(G)-1)(|H|-1)+t(G))/i].$$

If F is a graph with $|F| \geq (\chi(G)-1)(|H|-1)+t(G)$ and $\delta(F) \geq |F|-i$, then $F \to (G, H)$.

Proof. Let $\chi(G)=\chi$ and $t(G)=t$. Without loss of generality, we may assume that $G = K(t, n_2, \ldots, n_\chi)$. We prove that $F \to (G, H)$ by induction on χ.

If $\chi = 1$, the result is trivial. So assume $\chi \geq 2$ and assume that the result has been proved for $\chi - 1$. Let F be a graph with $|F| \geq (\chi-1)(|H|-1)+t$ and $\delta(F) \geq |F|-i$. Assume $F = F_1 \oplus F_2$ is a good factorization of F with respect to (G, H). Let H'' be the graph obtained from H' by subdividing edge ab with $|G|^2 i$ vertices. Since

$$\begin{aligned} r(G, H'') &\leq r(K_{|G|}, K_{|H''|}) \\ &\leq \binom{|G|+|H''|-2}{|G|-1} = \binom{|G|+|H'|+|G|^2 i-2}{|G|-1} \\ &\leq [((\chi-1)(|H|-1)+t)/i] \leq [|F|/i] \end{aligned}$$

and F contains a complete graph on $[|F|/i]$ vertices by Corollary 2, it follows that $F \to (G, H'')$. Thus F_2 contains H''.

Since F_2 contains H'' but does not contain H, select a graph H''' in F_2 which is obtained from H' by subdividing edge ab as many times as possible subject to the restrictions that $|H''| \leq |H'''| < |H|$. It follows that

$$\begin{aligned} |F - V(H''')| &\geq (\chi-1)(|H|-1)+t-(|H|-1) \\ &= (\chi-2)(|H|-1)+t. \end{aligned}$$

Let $F' = F - V(H''')$. Then, by the induction hypothesis, $F' \to (K(t, n_2, \ldots, n_{\chi-1}), H)$. Thus the intersection of F' and F_1 must contain a $K(t, n_2, \ldots, n_{\chi-1})$.

Let P be the long path in H''' with endpoints a and b, and let v be a vertex of F' which is adjacent in F_2 to the largest number of vertices of P. Let s be the number of vertices of P which are adjacent in F_2 to v. If $(s+i-1)(|G|-n_\chi) \leq |P|-n_\chi$, then there exist n_χ vertices of P which are adjacent in F_1 to each set of $|G|-n_\chi$ vertices of F'. But this gives a $K(t, n_2, \ldots, n_\chi)$ in F_1, a contradiction. So $(s+i-1)(|G|-n_\chi) > |P|-n_\chi$ and therefore

$$s-1 > (|P|-n_\chi)/(|G|-n_\chi)-i.$$

Let P' be the set of all vertices of P which are immediate predecessors of the vertices that are adjacent in F_2 to v (think of a as the initial endpoint of P when determining P'). Since $|H'''| \geq |H''|$, it follows that $|P| > |G|^2 i$ and therefore

$$|P'| \geq s-1 > (|G|^2 i - n_\chi - (|G|-n_\chi)i)/(|G|-n_\chi) \geq |G|\, i.$$

In addition, no two vertices of P' can be adjacent in F_2 by the maximality of H''', thus the subgraph of F_1 induced by $V(P')$ contains a $K_{|G|}$ by Corollary 2. But this gives a copy of G in F_1, a contradiction. Thus $F \to (G, H)$.

Theorem 4. *Let i be a positive integer, and let G and H' be fixed graphs with H' connected. Let $ab \in E(H')$, and let H be a graph obtained from H' by subdividing ab such that $|H| \geq |H'| + |G|^2 i$ and*

$$\binom{|G|+|H'|+|G|^2 i-2}{|G|-1} \leq [((\chi(G)-1)(|H|-1)+t(G))/i].$$

Then $r_i(G,H) = [((\chi(G)-1)(|H|-1)+t(G)-1)/i]+1.$

Proof. Let $\chi(G)=\chi$ and $t(G)=t$. Since $r(G,H)=(\chi-1)(|H|-1)+t$ (Burr [5]),

$$r_i(G,H) \geq [(r(G,H)-1)/i]+1 = [(\chi-1)(|H|-1)+t-1)/i]+1 = p.$$

Since $|K(i;p)| \geq (((\chi-1)(|H|-1)+t-1-(i-1))/i+1)i = (\chi-1)(|H|-1)+t$ and $\delta(K(i;p)) = |K(i;p)|-i$, it follows that $K(i;p) \to (G,H)$ by Theorem 3. Thus

$$r_i(G,H) = p = [((\chi-1)(|H|-1)+t-1)/i]+1.$$

Since $\chi(G)$ and $t(G)$ are easy to determine when G is a complete graph, star, path or cycle, Theorem 4 seems especially appropriate for these familiar graphs. First, however, we seek a lower bound for $|H|$. The inequality

$$|H| \geq ((|G|+|H'|+|G|^2 i-2)^{|G|-1}+1)i+1$$

is one such bound, since this value of $|H|$ implies that

$$\begin{aligned}\binom{|G|+|H'|+|G|^2 i-2}{|G|-1} &\leq (|G|+|H'|+|G|^2 i-2)^{|G|-1} \\ &\leq (|H|-1)/i-1 \\ &\leq ((\chi(G)-1)(|H|-1)+t(G)-i+1)/i \\ &\leq [((\chi(G)-1)(|H|-1)+t(G))/i].\end{aligned}$$

This bound and Theorem 4 give the following results.

Corollary 5. *If* $n \geq ((m+m^2 i)^{m-1}+1)i+1$, *then* $r_i(K_m, P_n) = [(m-1)(n-1)/i]+1.$

Corollary 6. *If* $n \geq ((m+m^2 i+1)^{m-1}+1)i+1$, *then*

$$r_i(K_m, C_n) = [(m-1)(n-1)/i]+1.$$

Corollary 7. *If* $n \geq ((m+(m+1)^2 i+1)^m+1)i+1$, *then*

$$r_i(K(1,m), P_n) = [(n-1)/i]+1.$$

Corollary 8. *If* $n \geq ((m+(m+1)^2 i+2)^m+1)i+1$, *then*

$$r_i(K(1,m), C_n) = [(n-1)/i]+1.$$

Corollary 9. *If* $n \geq ((m+m^2 i)^{m-1}+1)i+1$, *then* $r_i(P_m, P_n) = [(n+[\frac{1}{2}m]-2)/i]+1.$

Corollary 10. *If* $n \geq ((m+m^2i+1)^{m-1}+1)i+1$, *then*

$$r_i(P_m, C_n) = [(n+[\tfrac{1}{2}m]-2)/i]+1.$$

Corollary 11. *If* $n \geq ((m+m^2i)^{m-1}+1)i+1$, *then*

$$r_i(C_m, P_n) = \begin{cases} [(n+\tfrac{1}{2}m-2)/i]+1 & \text{if } m \text{ is even,} \\ [(2n-2)/i]+1 & \text{if } m \text{ is odd.} \end{cases}$$

Corollary 12. *If* $n \geq ((m+m^2i+1)^{m-1}+1)i+1$, *then*

$$r_i(C_m, C_n) = \begin{cases} [(n+\tfrac{1}{2}m-2)/i]+1 & \text{if } m \text{ is even,} \\ [(2n-2)/i]+1 & \text{if } m \text{ is odd.} \end{cases}$$

The lower bounds on n in the preceding corollaries are by no means sharp. In fact, we will use the techniques of Bondy and Erdös [3] to provide lower bounds in Corollaries 5 and 6 which are quadratic in m rather than exponential. As is customary, a path on the n vertices $\{x_1, \ldots, x_n\}$ will be denoted $(x_1, \ldots, x_n)$, while a cycle with the same vertices will be denoted $(x_1, \ldots, x_n, x_1)$. In addition, if G_1 and G_2 are subgraphs of G and $v \in V(G)$, we denote $|V(G_1) \cap N_{G_2}(v)|$ by $d_{G_1,G_2}(v)$. Now we present several preliminary results before establishing the main theorems.

Lemma 13 (Ore [9]). *If G is a graph on n vertices and the sum of the degrees of every pair of vertices of G is at least $n-1$, then G contains a P_n.*

Lemma 14. *Let G be a graph with $\delta(G) \geq |G|-i$ for some positive integer i and let $G = F_1 \oplus F_2$ be a good factorization of G with respect to (K_m, P_{n+1}). If F_2 contains a path P on n vertices, then $d_{P,F_2}(v) \leq im-1$ for each $v \notin V(P)$.*

Proof. Let $P = (x_1, \ldots, x_n)$ be a P_n in F_2. Assume there exists a vertex $v \notin V(P)$ such that $d_{P,F_2}(v) > im-1$. Let P' be the set of all vertices of P which are immediate predecessors of the vertices that are adjacent in F_2 to v. Since there is no P_{n+1} in F_2, $vx_1 \notin E(F_2)$ and therefore $|P'| \geq im$. In addition, no two vertices of P' can be adjacent in F_2, otherwise we have a P_{n+1} in F_2. Thus $F_1[V(P')]$ contains a K_m by Corollary 2, a contradiction. Hence $d_{P,F_2}(v) \leq im-1$ for each $v \notin V(P)$.

Lemma 15 (Erdös and Gallai [7]). *If G is a graph on n vertices such that $|E(G)| \geq \frac{1}{2}((m-1)(n-1)+1)$, then G contains a cycle of length at least m.*

Theorem 16. *Let G be a graph with $|G| \geq (m-1)(n-1)+1$ and $\delta(G) \geq |G|-i$. If $n \geq im^2-2m+3$, then $G \to (K_m, P_n)$.*

Proof. We proceed by induction on m. If $m=2$, the result follows easily by Lemma 13, since that lemma ensures that G contains a P_n.

Now assume $m > 2$ and assume that the theorem has been proved for $m-1$.

Let $n\geq im^2-2m+3$, and let G be a graph with $N=|G|\geq(m-1)(n-1)+1$ and $\delta(G)\geq|G|-i$. Assume $G=F_1\oplus F_2$ is a good factorization of G with respect to (K_m,P_n). Thus, by Turán's theorem,

$$|E(F_1)|\leq\frac{(m-2)(N^2-t^2)}{2(m-1)}+\binom{t}{2},$$

where $N\equiv t \bmod (m-1)$ and $0\leq t<m-1$. Since $t-m+1<0$ and $t\geq 0$, it follows that

$$|E(F_1)|\leq\frac{(m-2)N^2+t(t-m+1)}{2(m-1)}\leq\frac{(m-2)N^2}{2(m-1)}$$

and therefore

$$\begin{aligned}|E(F_2)|&\geq\frac{N(N-i)}{2}-|E(F_1)|\geq\frac{N(N-im+i)}{2(m-1)}\geq\frac{N[(m-1)(n-i-1)+1]}{2(m-1)}\\&>\tfrac{1}{2}N(n-i-1)\geq\tfrac{1}{2}((N-1)(n-i-1)+1).\end{aligned}$$

Thus, by Lemma 15, F_2 contains a path on at least $n-i$ vertices. Let P be a path in F_2 on r vertices, where r is as large as possible. Hence $n-i\leq r<n$. Let $S=V(G)-V(P)$. Since $r\leq n-1$, it follows that $|S|\geq N-(n-1)\geq(m-2)(n-1)+1$ and $\delta(G[S])\geq\delta(G)-r\geq|G[S]|-i$. Thus, by the induction hypothesis, $G[S]\to(K_{m-1},P_n)$. Hence $F_1[S]$ must contain a K_{m-1}. Let $x_1,\ldots,x_{m-1}$ denote the vertices of a K_{m-1} in $F_1[S]$. Since F_1 does not contain a K_m, no vertex of P can be adjacent in F_1 to each of $x_1,\ldots,x_{m-1}$. It follows that (at least) $r-(i-1)(m-1)$ vertices of P must be adjacent in F_2 to at least one of the vertices $x_1,\ldots,x_{n-1}$. But

$$\begin{aligned}r-(i-1)(m-1)&\geq(n-i)-(i-1)(m-1)\\&\geq(im^2-2m+3-i)-(i-1)(m-1)\\&=(im-1)(m-1)+1,\end{aligned}$$

and therefore some x_j must be adjacent in F_2 to at least im vertices of P contrary to Lemma 14. This contradiction implies $G\to(K_m,P_n)$.

Theorem 17. *If $n\geq im^2-2m+3$, then $r_i(K_m,P_n)=[(m-1)(n-1)/i]+1$.*

Proof. Since $r(K_m,P_n)=(m-1)(n-1)+1$ (Chvátal [6]),

$$r_i(K_m,P_n)\geq[(r(K_m,P_n)-1)/i]+1=[(m-1)(n-1)/i]+1=p.$$

Since $|K(i;p)|\geq(((m-1)(n-1)-(i-1))/i+1)i=(m-1)(n-1)+1$ and $\delta(K(i;p))=|K(i;p)|-i$, it follows that $K(i;p)\to(K_m,P_n)$ by Theorem 16. Hence

$$r_i(K_m,P_n)=p=[(m-1)(n-1)/i]+1.$$

Clearly, Theorem 17 establishes a sharper lower bound than the one given in Corollary 5. In addition, the following results will establish a better lower bound than the one given in Corollary 6.

Lemma 18 (Bondy [2]). *If $d_G(u)+d_G(v)\geq n$ for all vertices u and v such that $uv\notin E(G)$, then G contains cycles of all lengths l where $3\leq l\leq n$ or G is the complete bipartite graph $K(\frac{1}{2}n, \frac{1}{2}n)$.*

Lemma 19. *Let G be a graph with $\delta(G)\geq|G|-i$ for some positive integer i and let $G=F_1\oplus F_2$ be a factorization of G such that F_2 contains a C_n. If F_1 contains no K_m, then there is a cycle in F_2 of length c for some c such that $n-2im+3\leq c<n$ provided $n\geq 2im$.*

Proof. Let $C=(x_1,\ldots,x_n,x_1)$ be a C_n in F_2. Since there is no K_m in F_1, some pair of the im vertices $x_1,x_3,\ldots,x_{2im-1}$ must be adjacent in F_2 by Corollary 2, say x_j and x_k with $j<k$. Thus F_2 contains the cycle $(x_1,\ldots,x_j,x_k,x_{k+1},\ldots,x_n,x_1)$. It follows that F_2 contains a cycle of length c for some c such that $n-2im+3\leq c<n$.

Corollary 20. *Let G be a graph with $\delta(G)\geq|G|-i$ for some positive integer i and let $G=F_1\oplus F_2$ be a factorization of G such tha. F_2 contains a cycle C of length at least $n-i$. If F_1 contains no K_m, then there is a cycle in F_2 of length c for some c such that $n-2im+3\leq c\leq n$ provided $n\geq 2im$.*

Proof. If $n-i\leq|C|\leq n$, then C is a cycle in F_2 of length $|C|$ such that $n-2im+3\leq n-i\leq|C|\leq n$. Otherwise, C is a cycle in F_2 of length $|C|>n$. Then repeated application of Lemma 19 implies the existence of a cycle in F_2 of length c' such that $n-2im+3<c'\leq n$. We conclude that there is a cycle in F_2 of length c for some c such that $n-2im+3\leq c\leq n$.

Lemma 21. *Let G be a graph with $\delta(G)\geq|G|-i$ for some positive integer i and let $G=F_1\oplus F_2$ be a good factorization of G with respect to (K_m, C_{n+1}). If F_2 contains a cycle C of length n, then $d_{C,F_2}(v)\leq im-1$ for each vertex $v\notin V(C)$.*

Proof. Let $C=(x_1,\ldots,x_n,x_1)$ be a C_n in F_2. Assume there exists a vertex $v\notin V(C)$ such that $d_{C,F_2}(v)>im-1$. Let C' be the set of all immediate predecessors of vertices of C whch are adjacent in F_2 to v. No two vertices of C' can be adjacent in F_2, since otherwise we have a C_{n+1} in F_2. But then $F_1[C']$ contains a K_m by Corollary 2, a contradiction. Hence $d_{C,F_2}(v)\leq im-1$ for each vertex $v\notin V(C)$.

Theorem 22. *Let G be a graph with $|G|\geq(m-1)(n-1)+1$ and $\delta(G)\geq|G|-i$. If $n\geq\max\{2im, im^2+2im-i-m-2\}$, then $G\rightarrow(K_m, C_n)$.*

Proof. We proceed by induction on m. If $m=2$, the result follows easily by Lemma 18, since that lemma ensures that G contains a C_n.

Now assume $m>2$ and assume that the theorem has been proved for $m-1$. Let $n \geq \max\{2im, im^2+2im-i-m-2\}$, and let G be a graph with $N=|G| \geq (m-1)(n-1)+1$ and $\delta(G) \geq |G|-i$. Assume $G=F_1 \oplus F_2$ is a good factorization of G with respect to (K_m, C_n). Thus, by Turán's theorem,

$$|E(F_1)| \leq \frac{(m-2)(N^2-t^2)}{2(m-1)} + \binom{t}{2}.$$

where $N \equiv t \pmod{m-1}$ and $0 \leq t < m-1$. Since $t-m+1<0$ and $t \geq 0$, it follows (as in the proof of Theorem 16) that

$$|E(F_1)| \leq (m-2)N^2/2(m-1)$$

and therefore

$$|E(F_2)| \geq \tfrac{1}{2}((N-1)(n-i-1)+1).$$

Thus, by Lemma 15, F_2 contains a cycle of length at least $n-i$. Since there is no C_n in F_2, there is a cycle in F_2 of length c for some c such that $n-2im+3 \leq c < n$ by Corollary 20.

Let C be a cycle in F_2 of length c, where c is as large as possible subject to the bounds $n-2im+3 \leq c < n$. Let $S=V(G)-V(C)$. Since $c \leq n-1$, we have $|S| \geq (m-2)(n-1)+1$ and $\delta(G[S]) \geq |G[S]|-i$. Thus, by the induction hypothesis, $G[S] \to (K_{m-1}, C_n)$. Hence $F_1[S]$ must contain a K_{m-1}. Let $x_1, \ldots, x_{m-1}$ denote the vertices of a K_{m-1} in $F_1[S]$. Since F_1 does not contain a K_m, no vertex of C can be adjacent in F_1 to each of $x_1, \ldots, x_{m-1}$. It follows that (at least) $c-(i-1) \times (m-1)$ vertices of C must be adjacent in F_2 to at least one of the vertices $x_1, \ldots, x_{m-1}$. But

$$\begin{aligned} c-(i-1)(m-1) &\geq n-2im+3-(i-1)(m-1) \\ &\geq im^2+2im-i-m-2-2im+3-(i-1)(m-1) \\ &= im(m-1) \end{aligned}$$

and therefore some x_j is adjacent in F_2 to at least im vertices of C contrary to Lemma 21. This contradiction implies $G \to (K_m, C_n)$.

Theorem 23. *If* $n \geq \max\{2im, im^2+2im-i-m-2\}$, *then*

$$r_i(K_m, C_n) = [(m-1)(n-1)/i]+1.$$

Proof. Clearly $r_i(K_m, C_n) \geq r_i(K_m, P_n) = [(m-1)(n-1)/i]+1 = p$.

Since $|K(i;p)| \geq (m-1)(n-1)+1$ and $\delta(K(i;p)) = |K(i;p)|-i$, it follows that $K(i;p) \to (K_m, C_n)$ by Theorem 22. Hence $r_i(K_m, C_n) = p = [(m-1)(n-1)/i]+1$.

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